Reconstruction of a Signal Using the Spectrum-Reversal Technique

Takayuki Arai and Yuichi Yoshida

***Abstract*— Our procedure of real-zero conversion uses a spectrum-reversal technique to convert the information of a bandlimited signal to real zeros. We conducted a simple reconstruction experiment and showed that our proposed method is essentially equivalent to the conventional technique of sine-wave crossings.**

***Index Terms*—Bandlimited signals, level-crossing problems, minimum/maximum phase, poles and zeros, spectral-reversal technique.**

I. Introduction

A bandlimited (BL) signal, as a function of a complex variable, is an entire function of the exponential type and is determined by its zeros [1]. If the BL signal is a real signal, it has real zeros (RZs) or complex-conjugate zero (CZ) pairs. The RZs of the real signal can be observed as zero crossings of the waveform, whereas the CZ pairs are, in general, difficult to determine. In case the real signal has only RZs, the signal is called an RZ signal and is maximum phase (MaxP); on the other hand, the real signal, which has only CZ pairs, is minimum phase (MinP) [1]. In general, however, an arbitrary signal has both RZs and CZs.

There exist some techniques that transform a BL signal into an RZ signal, enabling the use of zeros as the signal's information source. A technique based on sine-wave crossings [2] is one such transformation. Nth-order differentiation [3] is another technique; this latter technique, however, has the drawback of emphasizing high-frequency components.

A technique proposed by Voelcker [1] for the inverse transformation from RZs to an RZ signal is called RZ interpolation (RZI). The RZI allows us to reconstruct the original signal from an RZ signal. A practical application of the RZI was presented by Sekey [3].

In this study, we propose an alternative procedure to transform a BL signal into an RZ signal using the spectrum-reversal technique. In Section II, we describe the characteristics of a spectrum reversal, which transforms MinP to MaxP and vice versa. We first review the background for a continuous-time signal in Section II-A. Then, we describe its new interpretation in Sections II-B and II-C. In Section III, we show that the combination of adding a DC and the spectrum-reversal technique converts a BL signal to MaxP since adding a DC essentially converts a BL signal to MinP. We then conducted a reconstruction experiment for a discrete-time signal. Finally, we show that our proposed method is essentially equivalent to the conventional technique: sine-wave crossings.

II. Spectrum-Reversal Technique

A. Minimum and Maximum Phase

An analytic signal $m(t)$ of a real signal $s(t)$ is represented by its envelope and phase as [4]

$$m(t) = s(t) + j\hat{s}(t) \tag{1}$$

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$$= |m(t)| \exp jo(t) \quad (2)$$

where $\hat{s}(t)$ is the Hilbert transform of $s(t)$.

The function $m(\zeta)$ is the entire function, where ζ is a complex time variable, i.e., $\zeta = t + j\sigma$ [6]. The relationship between $|m(t)|$ and $o(t)$ is determined by the zeros $\zeta_n = \tau_n + j\sigma_n$ of the function $m(\zeta)$ [1].

$$o'(t) = H[\ln' |m(t)|] + \sum_{nU} \frac{2r_n\sigma_n}{(\tau_n - t)^2 + \sigma_n^2} \quad (3)$$

where r_n is the order of the nth zero, and the index n_U denotes that the summation has been taken over all the zeros in the upper half of the ζ plane (UHP).

If there are no UHP zeros, the last term of the equation above becomes zero. In this case, $o'(t)$ is the minimum, and therefore, $m(\zeta)$ is called MinP. On the other hand, if all zeros are in the UHP, $o'(t)$ is a maximum, and therefore, $m(\zeta)$ is called MaxP [1].

B. Periodic BL Signal

If $m(t)$ is a periodic BL signal, it can be expressed by the Fourier series expansion.

$$m(t) = \sum_{k=0}^{N} c_k e^{jk\omega_0 t} \quad (4)$$

where c_k are the Fourier coefficients, and $\omega_0/2\pi$ is the fundamental frequency.

Let $z = e^{j\omega_0\zeta}$. The function $m(z)$ is then expressed by its zeros z_k $(k = 1, 2, \cdots, N)$ in factored form [5] as

$$m(z) = c_0 \prod_{k=1}^{N} \left(1 - \frac{z}{z_k}\right) \quad (5)$$

where c_0 is a real constant. By ignoring c_0, the zeros z_k are uniquely associated with $m(z)$.

Suppose that $m(z)$ has N zeros that are uniformly located on a circle with radius of $1/a$

$$z_k = a^{-1}e^{j2\pi k/N} \quad (6)$$

where $k = 0, 1, \cdots, N-1$.

If $0 < a < 1$, then $m(z)$ is MinP since the zeros ζ_k are located in the LHP of the ζ plane, or the zeros z_k are located outside the unit circle in the z plane. Expanding $m(z)$ and using its zeros

$$m(z) = c_0 \prod_{k=1}^{N} (1 - z_k^{-1} z) \quad (7)$$

$$= c_0 \prod_{k=1}^{N} (1 - ae^{-j2\pi k/N} \cdot z) \quad (8)$$

$$= c_0(1 - a^N z^N). \quad (9)$$

The signal $s(t)$, which is the real part of $m(t)$, is, therefore

$$s(t) = c_0(1 - a^N \cos N\omega_0 t). \quad (10)$$

Since $0 < a < 1$, $s(t)$ has no RZs.

Fig. 1 shows the case of $N = 8$. The solid line in Fig. 1(a) and the white small circles in Fig. 1(b) represent $s(t)$ and the zeros of $m(z)$, respectively.

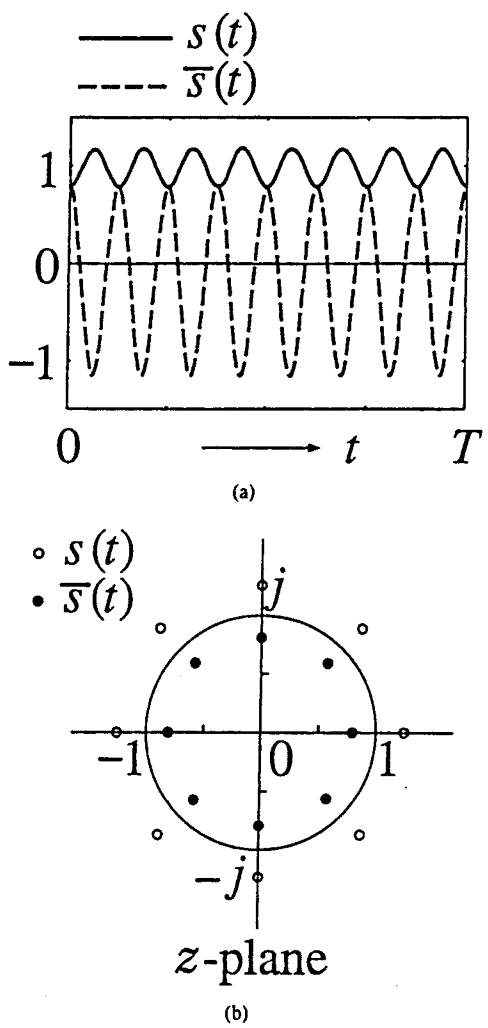


Fig. 1. Waveforms and zeros of $s(t) = 1 - a^N \cos N\omega_0 t$ with $a = 0.8$, $N = 8$, $\omega_0 = 2\pi/T$ (minimum phase) and the inverse (maximum phase). (a) Waveform and (b) its zeros in the z-plane.

C. Spectrum Reversal

Consider a spectrum-reversal signal $\overline{m}(t)$ of $m(t)$

$$\overline{m}(t) = \sum_{k=0}^{N} c_{N-k}^{*} e^{jk\omega_0 t} \quad (11)$$

where $*$ denotes complex conjugate. The signal $\overline{m}(z)$ is written in terms of the same zeros z_k

$$\overline{m}(z) = c_0 \prod_{k=1}^{N} \left(z - \frac{1}{z_k^{*}}\right). \quad (12)$$

Let z_k' denote zeros of the spectrum reversal signal $\overline{m}(z)$. Then

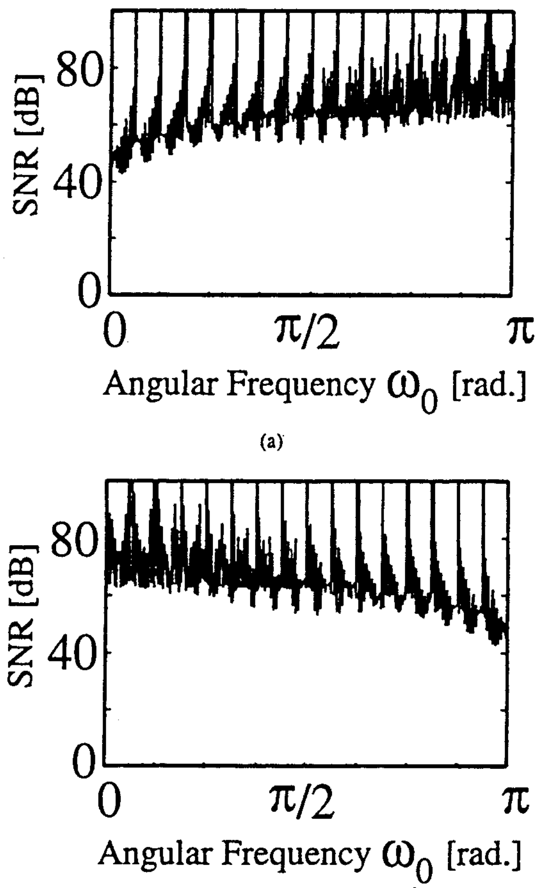
$$z_k' = \frac{1}{z_k^{*}} \quad (13)$$

$$= ae^{j2\pi k/N}. \quad (14)$$

Since the zeros ζ_k' are located in the UHP of the ζ-plane and the zeros z_k' are located inside the unit circle in the z-plane, $\overline{m}(z)$ is maximum phase. In this case, $s(t)$ becomes an RZ signal [1]. Then, $\overline{m}(z)$ is expanded using its zeros as

$$\overline{m}(z) = c_0(z^N - a^N) \quad (15)$$

$$= c_0 \prod_{k=1}^{N} (z - ae^{j2\pi k/N}). \quad (16)$$



(b)

Fig. 2. Signal-to-noise ratio of the reconstruction experiment. (a) Our proposed method. (b) Sine-wave crossings.

The signal $\bar{s}(t)$, which is the real part of $\overline{m}(t)$, is

$$\bar{s}(t) = c_0(\cos N\omega_0 t - a^N). \tag{17}$$

Since $0 < a < 1$, $\bar{s}(t)$ has RZs as shown in Fig. 1(a).

The dashed line in Fig. 1(a) represents $\bar{s}(t)$, and the filled small circles in Fig. 1(b) show the zeros of $\overline{m}(z)$. As seen in Fig. 1(b), the z_k and the z_k' are symmetrically located with respect to the unit circle.

III. New RZ Conversion Method

Sine-wave zero crossings [2] convert the signal information to RZs by adding a sinusoid with a larger amplitude and a higher frequency than the original signal. Let us denote $s_{DC}(t)$ as a periodic BL signal $s(t)$ with a DC component or a constant. If the added DC component is large enough, the zeros of $m_{DC}(z)$ will be located outside the unit circle. Therefore, $s_{DC}(t)$ becomes a MinP signal, and $\overline{m}_{DC}(z)$, which is the spectrum reversal signal of $m_{DC}(z)$, is MaxP. Finally, $\bar{s}_{DC}(t)$, which is the real part of $\overline{m}_{DC}(z)$, is an RZ signal. Since the added DC component will be converted to a high-frequency sinusoid by spectrum reversal, the proposed method is essentially equivalent to the sine-wave crossings.

For a discrete time signal, spectrum reversal is implemented by a π-radian spectrum shift, that is, by multiplying by $(-1)^n$ in the time domain. Each successive pair of samples of the signal $(-1)^n s_{DC}(n)$ always has the opposite sign; an RZ exists between them. If the length of $s(n)$ is N samples, $s(n)$ has N degrees of freedom. The signal $(-1)^n s_{DC}(n)$ also has N degrees of freedom and N RZs. Accordingly, the signal has no complex zeros. The reconstructed signal $\hat{s}(n)$ can be obtained by RZI from the RZs of $(-1)^n s_{DC}(n)$ after the π-radian spectrum shift and DC component removal.

We conducted a simple reconstruction experiment for a discrete-time signal. In this experiment,

$$s(n) = \cos \omega_0 n \tag{18}$$

where $0 \le \omega_0 \le \pi$, and

$$s_{DC}(n) = 1 + \alpha s(n) \tag{19}$$

where $\alpha = 0.01$, and $s_{DC}(n)$ effectively has a large DC component.

Fig. 2 shows the signal-to-noise ratio (SNR) derived from the original and the reconstructed signal as a function of ω_0. Fig. 2(a) shows the SNR of our proposed method, whereas Fig. 2(b) shows that of sine-wave crossings. As shown in Fig. 2, the two lines have an inverse relation to each other with respect to ω_0. In both cases, the SNR varies depending on ω_0 but is always larger than 40 dB.

IV. Conclusion

We have proposed a procedure of RZ conversion using a spectrum-reversal technique that is essentially equivalent to the technique with sine-wave crossings. To convert a BL signal into a MinP signal, we added a large constant to the original signal. The spectrum-reversal technique is applied to the MinP signal to obtain a MaxP signal. In practice, the spectrum reversal is implemented by the π-radian spectrum-shifted signal for a discrete time signal. The real part of the MaxP signal is an RZ signal, and all the information in the original signal is included in the RZs. This procedure shows that the information in any signal can be converted into zero-crossing information. We suggest here a new theoretical viewpoint based on the conventional theory such as that proposed by Voelcker [1].

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